Phase amplification in digital holography

Sri Rama Prasanna Pavani[1], Ethan F. Schonbrun[2]
[1]*Micro Optical – Imaging Systems Laboratory, Department of Electrical and Computer Engineering, University of Colorado at Boulder, USA*
[2]*School of Engineering and Applied Sciences, Harvard University, USA*
pavani@colorado.edu

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Abstract: We introduce phase amplification in digital holographic imaging systems by synthetically creating higher orders from interferograms recorded with linear detectors. Fundamental amplification limits, noise effects, and potential applications are discussed.

1. Introduction

Phase retardations of phase-only objects can be amplified in integral multiples by recording their image holograms in nonlinear media. Nonlinearly recorded holograms produce diffraction orders higher than the ± 1 orders that are typical of linearly recorded holograms. For phase-only objects, these higher orders contain multiples of the object's phase retardation [1]. Specifically, the n^{th} diffraction order of a nonlinear hologram carries a phase retardation that is n times amplified than the object's retardation. This phenomenon of phase amplification is attractive for the observation of weak phase objects. In digital holography, however, because the interferograms are typically recorded on linear detectors, their Fourier transforms do not have diffraction orders higher than ± 1. Consequently, phase amplification is not directly possible in digital holography.

In this paper, we present methods for achieving phase amplification in digital holography by synthetically obtaining higher orders from interferograms recorded on linear detectors. As in the case of nonlinearly recorded optical holograms, these higher orders contain multiples of object's phase. We discuss the factors that affect the maximum number of higher orders that can be obtained and the effect of noise in reconstruction from these higher orders.

2. Phase estimation

The interferogram I(x,y) of an object beam $E_o \exp\{-i[kz + \phi(x,y)]\}$ and an off-axis reference beam $E_o \exp\{-i[(k\cos\theta)z + (k\sin\theta)x]\}$ propagating at an angle θ with respect to z-axis is recorded with a detector at the plane $z = 0$. E_o is the constant amplitude of the object and reference beams, k is the wave number, and $\phi(x, y)$ is the object phase that needs to be estimated from I(x,y). I(x,y) can be compactly expressed as,

$$I(x,y) = 2{E_o}^2\{1 + \cos[\omega_o x - \phi(x,y)]\}, \tag{1}$$

where $\omega_o = 2\pi \sin(\theta)/\lambda$. Fourier transformation of this interferogram produces two conjugate symmetric first orders because of the interferogram's cosine carrier and a DC in the zero frequency region of the spectrum. The first orders are separated from DC by ω_o, which is the carrier frequency of the interferogram. When the spatial bandwidth of $\exp[i\phi(x,y)]$ is less than $2\omega_o$, the two first orders do not overlap with each other or with the DC, and therefore can be filtered out with a bandpass filter. The phase $\phi(x,y)$ encoded in the interferogram is the phase of the inverse Fourier transform of the filtered out +1 diffraction order [2]. Similarly, $-\phi(x,y)$ can be estimated from the inverse Fourier transform of the -1 diffraction order.

2. Phase amplification

The cosine carrier of the interferogram in equation (1) can only produce two first orders in the Fourier plane (Fig. 1a). When this interferogram is computationally modified to produce higher orders, it is seen that the higher orders carry multiples of the phase $\phi(x,y)$. In order words, the phase of the inverse Fourier transform of the n^{th} diffraction order is $n\phi(x,y)$. We describe two methods for obtaining higher orders: 1) binary amplitude hologram and 2) phase hologram. A straightforward method for obtaining higher orders is to create a binary amplitude hologram $I_b(x,y)$ by binarizing $I(x, y)$ such that $I_b(x,y) = 1$ if $I(x,y) > 2E_o^2$ and $I_b(x,y) = 0$ if $I(x,y) \leq 2E_o^2$. $I_b(x,y)$ can be equivalently be expressed as,

$$I_b(x,y) = 2E_o^2\{1 + \mathrm{square}[\omega_o x - \phi(x,y)]\} = 2E_o^2\left\{1 + \frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n}\cos[n(\omega_o x - \phi(x,y))]\right\} \tag{2}$$

Because of its square wave carrier, $I_b(x,y)$ produces $\pm1, \pm3, \pm5, .., \pm(2m-1)$ orders, where m is a positive integer. $\widetilde{I}_b(u,v)$, the Fourier transform of $I_b(x,y)$, can be directly computed by expressing (2) in terms of complex exponentials as,

$$I_b(x,y) = 2E_o^2\left(1 + \frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{1}{2n}\{\exp[-in(\omega_o x - \phi(x,y))] + \exp[in(\omega_o x - \phi(x,y))]\}\right), \tag{3}$$

$$\widetilde{I}_b(u,v) = 2E_o^2\left(\delta(u,v) + \frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{1}{2n}\left\{\widetilde{P}(u,v) *^n \widetilde{P}(u,v) * \delta(u - n\omega_o, v) + c.c\right\}\right) \tag{4}$$

where $\widetilde{P}(u,v)$ is the Fourier transform of $\exp[i\phi(x,y)]$, $*^n$ represents n times repeated convolution, and c.c means complex conjugate. From equation (3), the n^{th} order represents the Fourier transform of $\exp[in\phi(x,y)]$ and is separated from the DC by $n\omega_o$. In this method, since the diffraction efficiency decreases with n, the higher orders are weak (Fig. 1b).

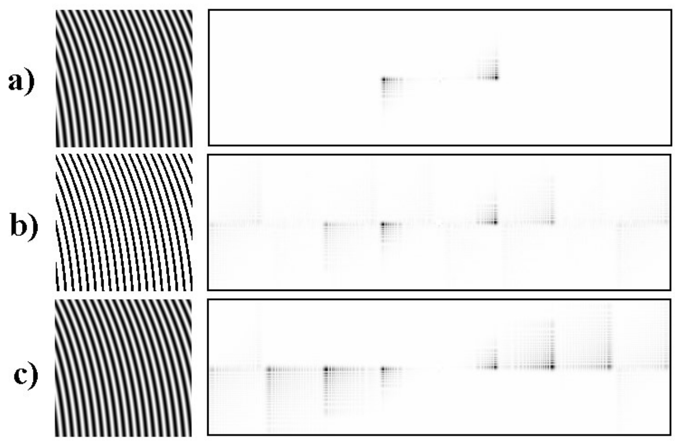


Figure 1. Digital holograms encoding cubic phase (left) and their dc-blocked Fourier transforms (right) for a) raw detected digital hologram $I(x,y)$, b) binary amplitude hologram $I_b(x,y)$, and c) phase hologram $I_p(x,y)$ with diffraction efficiency of 2^{nd} order maximized.

A yet another method of obtaining higher orders from $I(x,y)$ is to computationally synthesize a phase hologram $I_p(x,y)$, defined as, $I_p(x,y) = \exp[ikI(x,y)]$, where k is a constant. From Jacobi-Anger expansion,

$$\begin{aligned} I_p(x,y) &= \exp(i2kE_o^2)\exp\{i2kE_o^2\cos[\omega_o x - \phi(x,y)]\}, \\ &= \exp(i2kE_o^2)\sum_{n=-\infty}^{\infty} i^n J_n(2kE_o^2)\exp\{in[\omega_o x - \phi(x,y)]\}, \end{aligned} \tag{5}$$

where J_n is a Bessel function of the first kind. The Fourier transform of $I_p(x,y)$, $\widetilde{I}_p(x,y)$, can be expressed as,

$$\widetilde{I}_p(u,v) = \exp(i2kE_o^2)\left[\begin{array}{l} \sum_{n=-\infty}^{0} i^n J_n(2kE_o^2)\widetilde{P}(u,v) *^n \widetilde{P}(u,v) * \delta(u - n\omega_o, v) \\ + \sum_{n=1}^{\infty} i^n J_n(2kE_o^2)\widetilde{P}^*(-u,-v) *^n \widetilde{P}^*(-u,-v) * \delta(u + n\omega_o, v) \end{array}\right]. \tag{6}$$

From equation (5), similar to the case of a binary amplitude hologram, we see that the n^{th} order represents the Fourier transform of $\exp[in\phi(x,y)]$ and is separated from the DC by $n\omega_o$. The advantage of phase hologram is that the diffraction efficiency of any arbitrary order can be maximized by choosing an appropriate value for k.

Specifically, to maximize the diffraction efficiency of nth order, k is the number that maximizes the Bessel function $J_n(2kE_o^2)$.

3. Amplification limits

From equations (4) and (6), while the size of diffraction orders grow with the order number (because of convolution), their separation is always a constant (ω_o). Consequently, the orders higher than n_{max} physically overlap in the Fourier plane. We now express n_{max} in terms of the spatial bandwidth b of $\exp[i\phi(x,y)]$ and ω_o. From equations (4) and (6), the size of the n^{th} order in the Fourier plane is nb. For non-overlapping orders with highest n,

$$\begin{aligned} \omega_o &\geq \frac{n_{max}b}{2} + \frac{(n_{max}+1)b}{2} \\ n_{max} &\leq \frac{\omega_o}{b} - \frac{1}{2} \end{aligned} \tag{7}$$

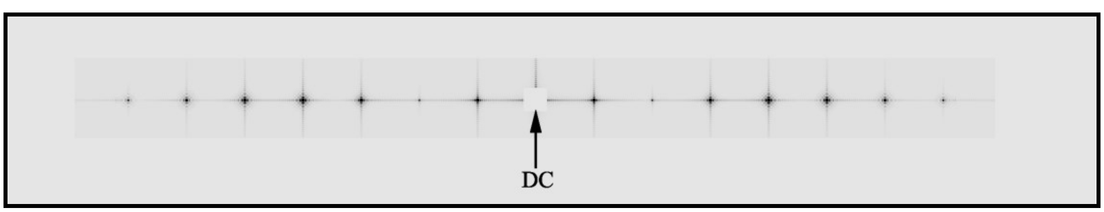
Another practical limit is related to the detector sampling frequency Ω_o. If the detector samples I(x,y) just at Nyquist, then the first orders are located at the edges of the Fourier plane. In such a case, any attempt to create higher orders would result in aliasing. Over-sampling is a requirement for phase amplification. Hence, the second limit for n_{max} is $n_{max} \leq \Omega_o/\Omega_{ny}$, where Ω_{ny} is the Nyquist frequency for I(x,y). Accounting for both limits together,

$$n_{max} \leq \min\left[\frac{\omega_o}{b} - \frac{1}{2}, \frac{\Omega_o}{\Omega_{ny}}\right]. \tag{8}$$

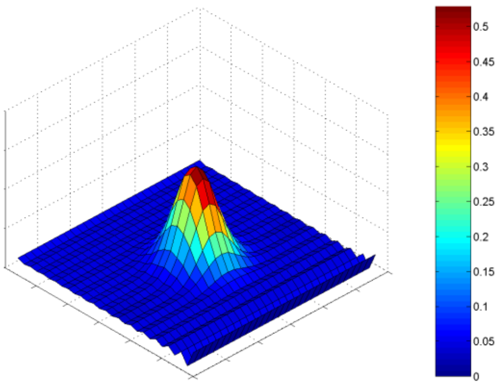
4. Noise effects

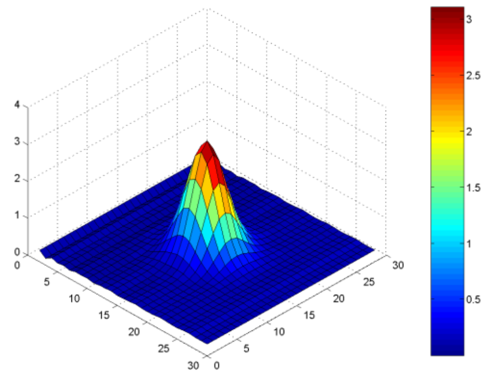
With an additive noise N(x,y), the noisy interferogram J(x,y) can be expressed as, J(x,y) = I(x,y) + N(x,y). In the binary amplitude hologram case, N(x,y) creates a binary noise which is particularly pronounced near the threshold $2E_o^2$. The spectrum of this binary noise adds to $\tilde{I}_b(u,v)$. However, in the case of the phase hologram, the spectrum of exp[iN(x,y)] convolves with $\tilde{I}_p(x,y)$. For a given SNR for J(x,y), the spectrum of phase hologram is noisier than that of the amplitude hologram. However, the phase hologram has the advantage of boosting the diffraction efficiency of a desired order, which means that the SNR of the desired order can be maximized in the phase hologram case.

5. Examples



(a)





(b) **(c)**

Figure 2 (a) FT of a phase hologram encoding a Gaussian phase, (b) reconstruction from 1st order, (c) reconstruction from the 6th order (showing 6 times the phase as the 1st order)

6. Acknowledgements

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7. References

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